THE SOLUTION STABILITY OF VAN DER POL'S EQUATION .

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** Abstract : The Van der Pol differential quation is solved by averaging method .
** Subjects: Vibration Mechanics , The Differential equations .

Introduction

This worksheet demonstrates Maple's capabilities in finding the graphical solution and dealing with the stability of the steady s tate solution of Van der Pol 's differential equation .

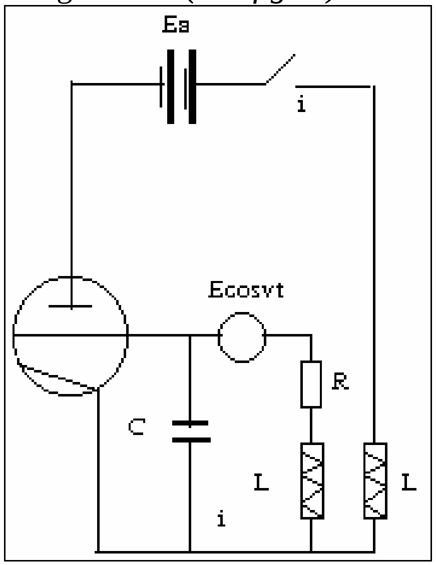
We consider the Van Der Pol differential equation :

$$x''+w^2x-m(1-lx^2)x'=0 \qquad (0)$$

Two topics that we will be in discussion are

- *Finding the steady state solution of this equation by averaging method .*
- *Estimating the stability of solution obtained .*

1 .Define the model of problem : We examine the effect of non-linear system under external force caused by the AC generator *(see fig 1.)*



$E\cos\upsilon t$: AC generator

C : Capacity

R : Resistance

L : Inductor

i : Current intensity

(fig 1.)

The differential equation of this model is given in the form

$$Lx''+(x-a)x'+\frac{x}{c}=E\cos vt \qquad (1)$$

After simplifying we obtain :

$$x''+k^2(x^2-1)x'+x=e\cos vt \qquad (2)$$

2 . Construct the algorithm .

Generally the Van Der Pol differential equation can be expressed by :

$$x'' + f(x, x')x' + g(x) = F(t) \qquad (3)$$

In the special case let $f(x, x') = m(1 - l x^2)$ and $g(x) = w^2 x$

the equation will be rewritten as : $x'' + w^2 x - m(1 - l x^2)x' = 0$

By using the transform $X = \sqrt{l}\, x; T = wt; m_X = \dfrac{m}{w}$ (4)

Then $x' = \frac{X'}{\sqrt{l}} w; x'' = \frac{X''}{\sqrt{l}} w^2$ and substitute these relations to (2) it follows :

$$\frac{w^2 X''}{\sqrt{l}} + \frac{w^2 X}{\sqrt{l}} - m_X w\left(1 - \frac{l X^2}{l}\right)\frac{wX'}{\sqrt{l}} = 0$$

Or $X'' + X - m_X(1 - X^2)X' = 0$ (5)

Thus we begin with : $x'' + x - m\ (1 - x^2)x' = 0$ (6)

To normalize this equation we find the solution which is expressed in the form

$x = a\cos(t + g)$

It is advantageous to write $j = t + g$ then the solution will be $x = a\cos j$ (7)

From the transform $x = a\cos j$, $x' = -a\sin j$.

We have $x'' = dx'/dt = -a'\sin j - a\cos j \,.\, j'$

By substituting to (6) , it gives

$$-a'\sin j\ -a\cos j\ .j\ '+a\cos j\ -m(1-a^2\cos^2 j\ ).(-a\sin j\ )=0 \qquad (8)$$

In the other hand $\quad dx/dt\ =\ x'\ =\ a'\cos j\ -\ a\sin j\ .j\ '\ =\ -a\sin j \qquad (9)$

Obviously we reach to the following system of differential equations

$$\begin{cases} -a'\sin j\ -a\cos j\ .j\ '=-m(1-a^2\cos^2 j\ ).a\sin j\ -a\cos j \\ a'\cos j\ -a\sin j\ .j\ '=-a\sin j \end{cases} \qquad (10)$$

```
> A:=Matrix([[-sin(phi),-a*cos(phi)], [cos(phi),-a*sin(phi)]]);
```

$$A := \begin{bmatrix} -\sin(\phi) & -a\cos(\phi) \\ \cos(\phi) & -a\sin(\phi) \end{bmatrix}$$

```
> u:=vector([-mu*a*(1-(a*cos(phi))^2)*sin(phi)-a*cos(phi),-a*sin(phi)]);
```

$$u := [-\mu\, a\,(1-a^2\cos(\phi)^2)\sin(\phi)-a\cos(\phi), -a\sin(\phi)]$$

```
> S:=simplify(linsolve(A,u));
```

$$S := [a\,(-1+\cos(\phi)^2)\,\mu\,(-1+a^2\cos(\phi)^2), \mu\sin(\phi)\cos(\phi)-\mu\, a^2\sin(\phi)\cos(\phi)^3+1]$$

```
> a(tt):=simplify(S[1]);
```

$$\mathrm{a}(tt) := a\,(-1+\cos(\phi)^2)\,\mu\,(-1+a^2\cos(\phi)^2)$$

```
> phi(tt):=simplify(S[2]);
```

$$\phi(tt) := \mu\sin(\phi)\cos(\phi)-\mu\, a^2\sin(\phi)\cos(\phi)^3+1$$

By using the symbols $\quad a(tt)\ =\ a'(t) \quad ,\quad f(tt)\ =\ f'(t) \qquad (11)$

Execute the averaging method for $a'(t)$ and $f'(t)$, we have

```
> a0(tt):=normal(int(mu*a*(1-a^2*cos(phi)^2)*sin(phi)^2,phi=gamma..gamma+2*Pi)/(2*Pi));
```

$$\mathrm{a0}(tt) := -\frac{1}{8}\mu\, a\,(-4+a^2)$$

```
> phi0(tt):=int(mu*a*(1-a^2*cos(phi)^2)*sin(phi)*cos(phi),phi=gamma..gamma+2*Pi)/(2*Pi);
```

$$\phi0(tt) := 0$$

The expressions of a' and $j\,'$ are calculated in the forms :

$$a'\ =\ m < \ \mathrm{a}(tt) := a\,(-1+\cos(\phi)^2)\,\mu\,(-1+a^2\cos(\phi)^2) >$$

$= \quad a0(tt) := -\frac{1}{8}\mu a(-4+a^2)$ (12)

$j' = m <\ \phi(tt) := \mu \sin(\phi)\cos(\phi) - \mu a^2 \sin(\phi)\cos(\phi)^3 + 1\ > \ = \ \phi0(tt) := 0$ (13)

The steady state solution occurs when $a0 = 0$ or $a0 = 2$

If $a0 = 0$ then $x = x' = 0$ this is a trivial solution (equilibrium) .

If $a0 = 2$ then $x = 2\cos j$ and $x' = -2\sin j$. (14)

The necessary and sufficient condition for solution stability includes

$a' = da/dt = Y(a)$ with $Y(ao) = 0$ and $Y'(ao) < 0$

```
> Psi(a):=a0(tt);
```

$\Psi(a) := -\frac{1}{8}\mu a(-4+a^2)$

```
> DaohamcuaPsi(a):=normal(diff(Psi(a),a));
```

$\text{DaohamcuaPsi}(a) := \frac{1}{2}\mu - \frac{3}{8}\mu a^2$

```
> print("Dao ham cua ham a'(t) la :","      a''(t)=    ",DaohamcuaPsi(a));
```

"Dao ham cua ham a'(t) la :", "   a''(t)=   ", $\frac{1}{2}\mu - \frac{3}{8}\mu a^2$

```
> subs(a=2,DaohamcuaPsi(a));
```

$-\mu$

```
> print("Gia tri cua a''(t) tai a = 2 la : a''(2) = ",subs(a=2,DaohamcuaPsi(a)));
```

"Gia tri cua a''(t) tai a = 2 la : a''(2) = ", $-\mu$

```
> subs(a=0,DaohamcuaPsi(a));
```

$\frac{1}{2}\mu$

```
> print("Gia tri cua a''(t) tai a = 0 la : a''(0) = ",subs(a=0,DaohamcuaPsi(a)));
```

"Gia tri cua a''(t) tai a = 0 la : a''(0) = ", $\frac{1}{2}\mu$

Thus if $ao = 0$, $a''(0) = \frac{1}{2}\mu$ then the solution is not stable .

If $ao = 0$, $a''(0) = -\mu$ then the solution is stable asymtotically.

Note : By solving the equation (12) for the vibration amplitude $a(t)$ (" slowly varying coefficients ") then finding the solution expression $x(t)$ of Van Der Pol differential equation , we get

```
> a0(tt);
```

$$-\frac{1}{8}\mu\, a\,(-4+a^2)$$

```
> diff_eq:= diff(a(t),t)=-mu*a(t)*(-4+(a(t)^2))/8;
```

$$diff\_eq := \frac{\partial}{\partial t}\mathrm{a}(t) = -\frac{1}{8}\mu\,\mathrm{a}(t)\,(-4+\mathrm{a}(t)^2)$$

```
> init_con:=a(0)=ao;
```

$$init\_con := \mathrm{a}(0) = ao$$

```
> biendo:=[dsolve({diff_eq,init_con}, {a(t)})];
```

$$biendo := \left[\mathrm{a}(t) = 2\frac{1}{\sqrt{1-\frac{e^{(-\mu t)}(ao^2-4)}{ao^2}}},\ \mathrm{a}(t) = -2\frac{1}{\sqrt{1-\frac{e^{(-\mu t)}(ao^2-4)}{ao^2}}}\right]$$

```
> biendo[1]:=simplify(biendo[1]);
```

$$biendo_1 := \mathrm{a}(t) = 2\frac{1}{\sqrt{-\frac{-ao^2+e^{(-\mu t)}ao^2-4\,e^{(-\mu t)}}{ao^2}}}$$

(accepted)

```
> biendo[2]:=simplify(biendo[2]);
```

$$biendo_2 := \mathrm{a}(t) = -2\frac{1}{\sqrt{-\frac{-ao^2+e^{(-\mu t)}ao^2-4\,e^{(-\mu t)}}{ao^2}}}$$

(eliminated)

```
> x:=biendo[1]*cos(phi);
```

$$x := \cos(\phi)\,\mathrm{a}(t) = 2\,\frac{\cos(\phi)}{\sqrt{-\dfrac{-ao^2 + \mathbf{e}^{(-\mu\,t)}\,ao^2 - 4\,\mathbf{e}^{(-\mu\,t)}}{ao^2}}}$$

with $j = t + g$. (15)

```
> x:=t->2*cos(t + gamma)/sqrt((a^2-a^2*exp(-mu*t)+4*exp(-mu*t))/a^2);
```

$$x := t \to 2\,\frac{\cos(t+\gamma)}{\sqrt{\dfrac{a^2 - a^2\,\mathbf{e}^{(-\mu\,t)} + 4\,\mathbf{e}^{(-\mu\,t)}}{a^2}}}$$

The graphical solution of Van Der Pol's equation :

```
> a:=0.5;
```

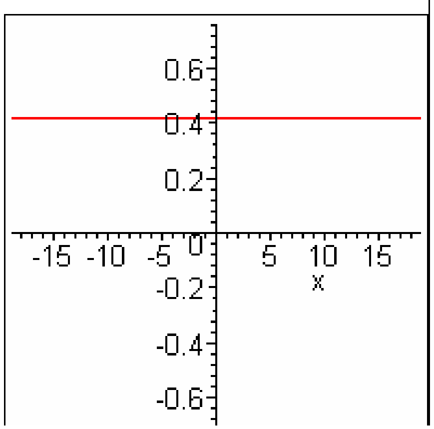
$$a := .5$$

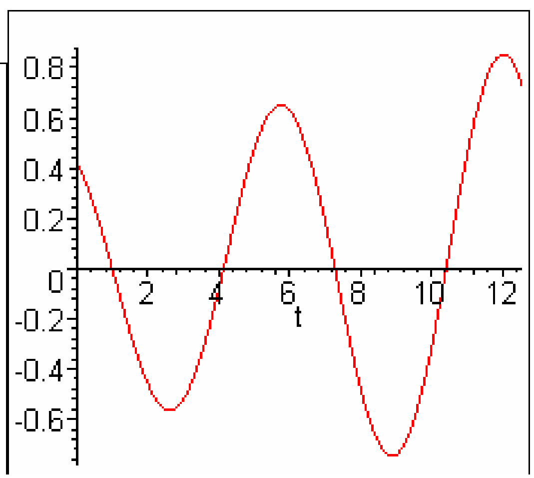
```
> y:=t->2*cos(t + gamma)/sqrt((a^2-a^2*exp(-0.1*t)+4*exp(-0.1*t))/a^2);
```

$$y := t \to 2\,\frac{\cos(t+\gamma)}{\sqrt{\dfrac{a^2 - a^2\,\mathbf{e}^{(-.1\,t)} + 4\,\mathbf{e}^{(-.1\,t)}}{a^2}}}$$

```
> plot(y(t),t=0..4*Pi);

> animate(2*cos(x*t + gamma)/sqrt((a^2-a^2*exp(-0.1*t)+4*exp(-0.1*t))/a^2),x=-6*Pi..6*Pi,
t=0..10);
```

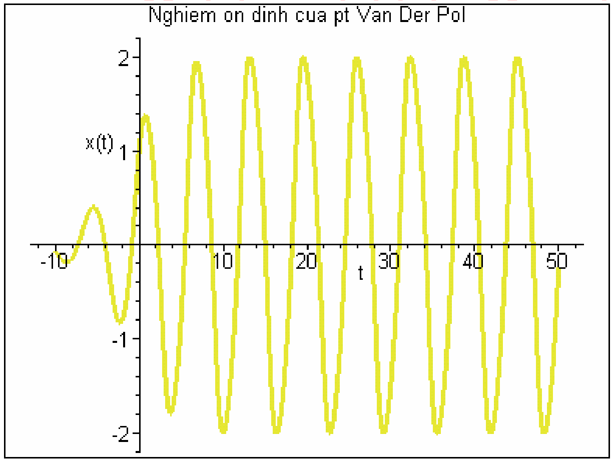




Use graphical method to consider the solution stability of Van Der Pol's differential equation :

```
> with(DEtools):mu:=0.5;
```

```
> DEplot({(D@@2)(x)(t)+x(t)-mu*(1-x(t)^2)*D(x)(t)=0},{x(t)},t=-
10..50,[[x(0)=1,D(x)(0)=1]],stepsize=0.05,title=`Nghiem on dinh cua pt Van Der Pol`);
```



3 . Conclusion .

From graphical results , we reach to conclusion that the steady state solution stability of Van Der Pol diffential equation must be precise
and estimating the property of solution obtained is very necessary .
As presented above , we might also use the normalization to (2) by determining the non-trivial solution in the form

$$\begin{cases} x = M\cos kt + N\sin kt + x^* \\ x' = -kM\sin kt + kN\cos kt + x^{*\prime} \end{cases} \qquad (16)$$

with $k = 1$

$$dx'/dt = x'' = -M'\sin t - M\cos t + N'\cos t - N\sin t + x^{*\prime\prime}$$

Otherwise

$$dx/dt = x' = M'\cos t - M\sin t + N'\sin t + N\cos t + x^{*}{}'$$

$$= -M\sin t + N\cos t + x^{*}{}'$$

Substitute x " to (6) after simplifying it follows :

$$\begin{cases} M'\cos t + N'\sin t = 0 \\ -M'\sin t + N'\cos t = m[1-(M\cos t + N\sin t)^2].[-M\sin t + N\cos t] \end{cases}$$

```
> A:=Matrix([[cost,sint], [-sint,cost]]);
```

$$A := \begin{bmatrix} cost & sint \\ -sint & cost \end{bmatrix}$$

```
> f:=vector([[0], [mu*F]]);
```

$$f := [[0], [\mu F]]$$

```
> V:=linsolve(A,f);
```

$$V := \left[\frac{sint\,[-\mu F] - [0]\,cost}{sint^2 + cost^2}, -\frac{sint\,[0] + [-\mu F]\,cost}{sint^2 + cost^2} \right]$$

We rewrite the expressions : $M' =$ $sint\,[-\mu F]$

$N' =$ $[\ \mu F]\,cost$ (17)

With

$$F = [1-(M\cos t + N\sin t)^2].[-M\sin t + N\cos t]$$

Use averaging method for (17) we get :

$$M' = m < -Fsint > = -\frac{1}{8} M (4 - M^2 + 2N^2 - 3N)$$

$$N' = m < Fcost > = -\frac{1}{8} N (-4 - 5M^2 + N)$$

(18)

The steady state solution exists when $M = 0$ and $N = 0$ (trivial solution)

Or $M = 2$ or $M = -2$ and $N = 0$

Thus $x = 2cost$, $x = -2cost$

If $M = 0$ and $N = 0$ or $N = 4$ Then $x = 4sint$

The steady state solutions can be formed generally :

$$\begin{cases} x = 2\cos t + 4\sin t \\ x = -2\cos t + 4\sin t \end{cases}$$

But these forms are equivalent to $x = 2cos\, j$ [see (14)]

Next we begin with Krylov –_Bogoliubov approximate method for the Van der Pol's differential equation

$$x'' + w^2 x - m(1 - l x^2)x' = 0$$

with $f(x, x') = -(1 - l x^2)x'$, and m is a small constant .

The solution will be estimated by $x = r(t)\cos j(t)$

By taking the first order approximate terms (neglecting the second and third order errors of constant m) we can find the amplitude $r(t)$ and the global phase function $j(t)$ from the following system

APPROVED

By CO HONG TRAN at 1:38 pm, Jun 24, 2009

$$\begin{cases} r'(t) = \dfrac{m}{2pw}\displaystyle\int_0^{2p} f(r\cos u, -rw\sin u)\sin u\,du = -\dfrac{r}{2}a_1(r) \\ j'(t) = w + \dfrac{m}{2prw}\displaystyle\int_0^{2p} f(r\cos u, -rw\sin u)\cos u\,du = \sqrt{a_2(r)} \end{cases} \qquad (19)$$

If the initial condition $r(0) = ro$ is satisfied then the solution of (15) will be equivalent to the solution of second order linear differential equation

$$x''(t) + a_1(r_o)x'(t) + a_2(r_o)x(t) = 0 \qquad (20)$$

with the error of estimation based on the order of m^2

The first order approximate is noticeable in the case of periodic vibration , because the equivalent linear differential equation gives us the accumulation and dissipation of energy based on vibration period which we might obtain from the given non-linear differential equation . Therefore it is useful to apply the equivalent second linear differential equation to observe the non-linear resonant phenomenon .

REFERENCES

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[2] **Korn . G , Korn . T .,** *So tay toan hoc* , Trans Vietnamese , NXB DH-THCN , Hanoi 1978